VERBAL SUBGROUPS OF HYPERBOLIC GROUPS HAVE INFINITE WIDTH

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ABSTRACT. Let G be a non-elementary hyperbolic group. Let w be a proper group word. We show that the width of the verbal subgroup $w(G) = \langle w[G] \rangle$ is infinite. That is, there is no such $l \in \mathbb{Z}$ that any $g \in w(G)$ can be represented as a product of $\leq l$ values of w and their inverses. As a consequence, we obtain the same result for a wide class of relatively hyperbolic groups.

1. INTRODUCTION

In this paper we show that every non-elementary hyperbolic group G is *verbally parabolic*, i.e., every proper verbal subgroup of G has infinite width. This immediately implies that every group that have a non-elementary hyperbolic image is also verbally parabolic, thus giving a wide class of verbally parabolic groups.

Let $F(X)$ be a free group with basis X and $W \subseteq F(X)$ a subset of $F(X)$. An element $g \in G$ is called a W*-element* if g is the image in G of some word $w \in W$ under a homomorphism $F(X) \to G$. By $W[G]$ we denote the set of all W-elements in G (the *verbal* set defined by W). The set $W[G]$ generates the *verbal* subgroup $W(G)$. The verbal subgroups of groups were intensely studied in group theory especially with respect to relatively free groups and varieties of groups (see the book [31] for details). If W is finite then $W(G)$ is equal to $w(G)$ for a suitable single word w. From now on we always assume that W is just a singleton $W = \{w\}$ and refer to the related verbal set and the subgroup as $w[G]$ and $w(G)$.

The w*-width* (or w*-length*) $l_w(g)$ of $g \in w(G)$ is the least natural number n such that g is a product of n w-elements in G or their inverses. The *width* l_w of w, as well as the w*-width* of the verbal subgroup $w(G)$, is defined by $l_w = \sup\{l_w(g) \mid g \in w(G)\}$, so it is either a natural number or ∞.

Finite width of verbal subgroups plays an important part in several areas of groups theory: finite groups, profinite and algebraic groups, polycyclic and abelian-by-nilpotent groups.

In finite groups one of the earliest questions on verbal width goes back to the Ore's paper [35] where he asked if the commutator length (the $[x, y]$-length) of every element in a non-abelian finite simple group is equal to 1 (*Ore Conjecture*). Only recently the conjecture was established by Liebeck,

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O'Brian, Shalev and Tiep [18]. For other recent spectacular results on verbal width in finite simple groups we refer to the papers [17], [43].

In profinite, as well as algebraic, groups the interest to verbal width comes from a fact that verbal subgroups of finite width are closed in profinite, correspondingly Zariski, topology. This shows up in the proof of Serre's result [42]: if G is a finitely generated pro-p group then every subgroup of finite index is open. Serre raised the question if the above result holds for arbitrary finitely generated profinite groups. In 2007 this conjecture was settled in affirmative by Nikolov and Segal [32]. Their proof is based in large on establishing uniform bounds on verbal width in finite groups. We refer to the book [41] on verbal width in profinite and algebraic groups and to a recent survey [33].

The study of verbal sets and subgroups in infinite groups was initiated by P. Hall [15, 16]. In 1960's several fundamental results on verbal width in infinite groups were obtain in his school: Stroud proved (unpublished) that all finitely generated abelian-by-nilpotent groups G are *verbally elliptic*, i.e., every verbal subgroup $w(G)$ of G has finite w-width; meanwhile Rhemtulla showed in [38] that every not infinite dihedral free product $G = A * B$ of non-trivial groups A and B is verbally parabolic. We refer again to the book [41] for the history and detailed modern proofs.

Shortly afterwards, Hall's approach to verbal and marginal subgroups gave rise to an interesting research in Malcev's algebraic school in Russia. Firstly, Merzlyakov [22] showed that linear algebraic groups are verbally elliptic, and then Romankov [40] (independently of Stroud and using different means) showed that finitely generated virtually nilpotent groups are all verbally elliptic; as well as all virtually polycyclic groups (again, we refer to [41] for concise proofs, generalizations and related new developments). Since Malcev's school was keen on model-theoretic problems in groups, it was quickly recognized there that verbal subgroups of finite width are first-order definable in the group, so they provide a powerful tool in the study of elementary theories of verbally elliptic groups, in particular, nilpotent groups. Since then the elementary theories of finitely generated nilpotent groups and groups of K-points (K is a field) of algebraic nilpotent groups were extensively studied [19, 9, 26, 27, 28, 23, 24, 25]. Nowadays, it is known precisely which finitely generated nilpotent groups are elementarily equivalent [34]; or what are arbitrary groups (perhaps, non-finitely generated) which are elementarily equivalent to a given unipotent K-group [23, 24], or a given unitriangular group $UT(n, \mathbb{Z})$ [3], or a given finitely generated free nilpotent group [30].

In the opposite direction, Remtulla's remarkable proof of verbal parabolicity of non-abelian free groups and non-trivial free products gave rise to several interesting generalizations. In [11] Grigorchuk studied bounded cohomologies of group constructions, and proved that verbal width of the commutator subgroup in a wide class of amalgamated free products and HNN extensions is infinite. Bardakov showed that HNN extensions with proper

associated subgroups and one-relator groups with at lest three generators are verbally parabolic [1]; as well as braid groups B_n [2]. For amalgamated free products Dobrynina improved on the previous partial results by showing that $A *_U B$ is verbally parabolic, provided $U \neq A$ and $UbU \neq Ub^{-1}u$ for some $b \in B$ [7].

In this paper we show that many groups that have some kind of "negative curvature" are virtually parabolic. The main result in its pure form (Theorem 1) states that every non-elementary (i.e., non virtually cyclic) hyperbolic group G is verbally parabolic. As was mentioned above this implies that every group that have a non-elementary hyperbolic image is also verbally parabolic. In particular, the following groups are verbally parabolic: non-abelian residually free groups, pure braid groups, non-abelian right angled Artin groups — all of them have free non-abelian quotients (so these are corollaries of the original Remtulla's result on free groups). Less obvious examples include many relatively hyperbolic groups, e.g., non-elementary groups hyperbolic relative to proper residually finite subgroups [37]. Thus, the fundamental groups of complete finite volume manifolds of pinched negative curvature, $CAT(0)$ groups with isolated flats, groups acting freely on $\mathbb{R}^n$–trees are verbally parabolic.

The results above show that in the presence of negative curvature the verbal width is not a very sensitive characteristic of verbal subgroups. In this case it seems more convenient to consider a more smooth *stable* w-length, which is defined for an element $g \in w(G)$ as the limit $\lim_{n\to\infty} \frac{l_w(g^n)}{n}$. It is known that the stable commutator length relates to an L^1 filling norm with rational coefficients, introduced by Gromov in [12], see also Gersten's paper [10]. In [14] Gromov studied stable commutator length and its relation with bounded cohomology. We refer to a book [5] by Calegary on stable length of elements in groups. Interestingly, Calegari showed in [6] that if a group G satisfies a non-trivial law then the stable commutator length is equal to 0 for every element from $[G, G]$. So it seems the verbal width and stable length operate nicely in very different classes of groups.

We formulate a few open problems on verbal width in infinite groups. The first one, if true, would make proofs of many results easier.

Problem 1. *Is it true that finite extensions of virtually parabolic groups are virtually parabolic?*

Notice, that in general, verbal parabolicity is not a geometric property. Indeed, there is a group H with finite commutator width (the commutator width is equal to 1 in H), but some of its finite extensions has infinite commutator width [41, Exercise 3.2.2], so infinite width is not preserved by quasi-isometries. The second problem is an attempt to clarify if interesting groups without free subgroups can be verbally parabolic. It seems all known verbally parabolic groups have free non-abelian subgroups.

Problem 2. *Is Thompson group V verbally parabolic?*

In general, verbal sets $w[G]$ contain a lot of information about the group G. For example, the Membership Problem (MP) to the set $w[G]$ in G is equivalent of solving in G the homogeneous equations of the type $w(X) = g$, where $g \in G$. Notice, that the Diophantine Problem (of solvability of arbitrary equations) is decidable in a free group F [21], so the verbal sets are decidable. However, the actual algorithmic or formal language theoretic complexity of verbal sets in F is unknown.

Problem 3. *Is it true that proper verbal subsets of a non-elementary hyperbolic group are non-rational?*

It is known that proper verbal set in free non-abelian groups are non-rational [29]. Remtulla's technique from [38] played an important part in the proof of this result.

2. Preliminaries

2.1. **Verbal sets and subgroups.** Let $F = F(X)$ be a free group with basis $X = \{x_1, \ldots, x_k, \ldots\}$, viewed as the set of reduced words in $X \cup X^{-1}$ with the standard multiplication. Throughout we use notation and definitions from Introduction.

By $w[G]^l$ we denote the set of elements $g \in G$ for which there exist $g_{ij} \in G$ such that

$$g = w^{\pm 1}(g_{11}, g_{12}, \ldots) \cdot w^{\pm 1}(g_{21}, g_{22}, \ldots) \cdots w^{\pm 1}(g_{l1}, g_{l2}, \ldots).$$

A word $w = w(x_1, x_2, ..., x_n) \in F$ is termed *proper* if there exist groups G and H such that $w[G] \neq 1$ and $w[H] \neq H$, in fact, in this case $1 \neq w[G \times H] \neq G \times H$.

Any element $w \in F$ can be written as a product

$$w = x_1^{t_1} x_2^{t_2} ... x_n^{t_n} w',$$

where $w' \in [F, F]$, $t_1, \ldots, t_n \in \mathbb{Z}$. Since the exponents $t_1, \ldots, t_n$ depend only on the element w the number $e(w) = gcd(t_1, \ldots, t_n)$ is well-defined (here we put $e(w) = 0$ if $t_1 = \ldots = t_n = 0$). If $e(w) = 0$ then we refer to w as a *commutator word*. A non-trivial commutator word is obviously proper (for instance, it is proper in F). If $e(w) > 0$ then there exist integers $r_1, r_2, ..., r_n$ such that $\sum_{i=1}^{n} r_i t_i = e$, so for an arbitrary group G and an element $g \in G$ one has $w(g^{r_1}, g^{r_2}, ..., g^{r_n}) = g^e$. In particular, $e(w) = 1$ implies that $w[G] = G$ for every group G, so w is not proper. If $e(w) > 1$ then w is proper, which can be seen in an infinite cyclic group. The argument above shows that, a non-trivial word $w \in F$ is proper if and only if $e(w) \neq 1$. The following is the main result of the paper.

Theorem 1. *Every non-elementary hyperbolic group G is verbally parabolic, i.e., every proper verbal subgroup of G has infinite width.*

2.2. **Hyperbolic geometry.** In this section we recall some known facts about hyperbolic metric spaces (see [13, 8, 4] for general references).

Let M be a metric space with distance $\mathrm{dist}(x,y)$ which we also denote by $|x-y|$. Fix a constant $K \in \mathbb{N}$. Recall, that two paths p and q in M *K-fellow travel* if $\mathrm{dist}(p(t), q(t)) \leq K$ for all $t \geq 0$ [8]. They *asynchronously K-fellow travel* if there are non-decreasing proper continuous functions $\varphi, \psi : [0,\infty) \to [0,\infty)$ such that $\mathrm{dist}(p(\varphi(t)), q(\psi(t))) \leq K$ for all $t \geq 0$.

For a subset $Y \subseteq M$ and a number $H \in \mathbb{N}$ by $\mathcal{N}_H(Y)$ we denote the closed H-neighborhood of Y. As usual, two subsets Y_1 and Y_2 of M are *K-close* if the Hausdorff distance between them does not exceed K, i.e., $Y_1 \subseteq \mathcal{N}_K(Y_2)$ and $Y_2 \subseteq \mathcal{N}_H(Y_1)$.

Evidently, if paths p, q asynchronously K-fellow travel then they are K-close. The converse is true for quasigeodesic paths.

Lemma 1. *Suppose p, q are K-close (λ, ε)-quasigeodesics in a δ-hyperbolic geodesic metric space $\mathcal{H}$ that originate from points p_0, q_0, respectively, such that $|p_0 - q_0| \leq K$. Then p and q asynchronously K'-fellow travel for some constant travel, where $K' = K'(\delta, \lambda, \varepsilon, K)$.*

Proof. It is shown in the proof of Lemma 7.2.9 of [8] that the required statement holds in presence of so-called departure function D for p, q. Recall that for a path $\alpha : [0,T] \to \mathcal{H}$, function $D : \mathbb{R} \to \mathbb{R}$ is called a departure function if $|\alpha(s) - \alpha(s+t)| \geq r$ whenever $s, t+s \in [0,T]$ and $t \geq D(r)$. Note that in case of (λ, ε)-quasigeodesics, $D(r) = \lambda r + \lambda\varepsilon$ is a departure function. □

Lemma 2. *Let $\mathcal{H}$ be a δ-hyperbolic geodesic metric space. Let p be a geodesic path, and q be a (λ, ε)-quasigeodesic path in $\mathcal{H}$ joining points P, Q and P, S, respectively. Suppose $H \geq 0$ is such that $|QS| \leq H$. Then there exists $K = K(\delta, \lambda, \varepsilon, H) \geq 0$ such that p, q asynchronously K-fellow travel.*

Proof. It is known (see, for example, [13, Section 7.2] or [4, Theorem III.1.7]) that such paths p, q are K-close for some $K = K(\delta, \lambda, \varepsilon + H)$. Now, the asynchronous K'-fellow travel property follows from Lemma 1. □

Lemma 3. *Let $\mathcal{H}$ be a δ-hyperbolic geodesic metric space. Let p, q be two (λ, ε)-quasigeodesic paths in $\mathcal{H}$ joining points P_1, P_2 and Q_1, Q_2, respectively. Suppose $H \geq 0$ is such that $|P_1Q_1| \leq H$ and $|P_2Q_2| \leq H$. Then there exists $K = K(\delta, \lambda, \varepsilon, H) \geq 0$ such that p, q asynchronously K-fellow travel.*

Proof. Join points P_1 and Q_2 with a geodesic s. Then by Lemma 2 p, s and s, q asynchronously K-fellow travel for some K, so they are K-close. Hence p and q are $2K$-close, and the result follows from Lemma 1. □

3. Big Powers Products

The following notation is in use throughout the whole paper. Let G be a non-elementary hyperbolic group with a generating set A. Put $\mathcal{A} = A \cup A^{-1}$ and denote by $\mathcal{A}^*$ the free monoid generated by $\mathcal{A}$. For a word $h \in \mathcal{A}^*$ by $\overline{h}$

we denote the image of h under the canonical epimorphism $\mathcal{A}^* \to G$. We say that a word h is (quasi)geodesic in G if the corresponding path in Cayley graph of G relative to $\mathcal{A}$ is (quasi)geodesic. For $g \in G$, by $|g|$ we denote the geodesic length of g. Let δ be a hyperbolicity constant of G relative to A.

For an element $g \in G$ of infinite order, $E(g)$ denotes the unique maximal elementary subgroup of G containing g.

The following lemma is due to Ol'shanski [36].

Lemma 4. [36, Lemma 2.3] *Let $h_1, \ldots, h_\ell$ be words in $\mathcal{A}$ representing elements $\overline{h}_1, \ldots, \overline{h}_\ell$ of infinite order in G for which $E(\overline{h}_i) \neq E(\overline{h}_j)$ for $i \neq j$. Then there exist constants $\lambda = \lambda(h_1, \ldots, h_\ell) > 0$, $\varepsilon = \varepsilon(h_1, \ldots, h_\ell) \geq 0$, and $N = N(h_1, \ldots, h_\ell) > 0$ such that any word of the form*

$$h_{i_1}^{m_1} h_{i_2}^{m_2} \cdots h_{i_s}^{m_s},$$

where $i_k \in \{1, \ldots, \ell\}$, $i_k \neq i_{k+1}$ for $k = 1, \ldots, s-1$, and $|m_k| > N$ for $k = 2, \ldots, s-1$, is (λ, ε)-quasigeodesic in G.

Elements $g_1, g_2 \in G$ are termed *commensurable* if $h^{-1} g_1^{n_1} h = g_2^{n_2}$ for some $h \in G$ and $n_1, n_2 \in \mathbb{Z}, n_1, n_2 \neq 0$.

Proposition 1. *Let G be a non-elementary hyperbolic group. Let b, f_0, f_1 be words in $\mathcal{A}$ such that elements $\overline{b}, \overline{f}_0, \overline{f}_1 \in G$ are pairwise non-commensurable. Then there exist a number $M > 0$ and constants λ, ε that depend on b, f_0, f_1, such that the following condition holds. Let g be a word of the form*

$$g = g_1^{m_1} g_2^{m_2} \cdots g_k^{m_k},$$

where $g_i \in D = \{b^{\pm 1}, f_0^{\pm 1}, f_1^{\pm 1}\}$, $m_i > M$ $(i = 1, \ldots, k)$, and $g_i \neq g_{i+1}^{\pm 1}$ $(i = 1, \ldots, k-1)$. Then g is (λ, ε)-quasigeodesic in G.

Proof. The statement immediately follows from Lemma 4. □

Replacing b, f_0, f_1 with b^M, f_0^M, f_1^M we get the following corollary.

Corollary 1. *Let G be a non-elementary hyperbolic group. Let b, f_0, f_1 be words in $\mathcal{A}$ such that elements $\overline{b}, \overline{f}_0, \overline{f}_1 \in G$ are pairwise non-commensurable. Then there exist constants λ, ε that depend on b, f_0, f_1, such that the following condition holds. Let g be a word of the form*

$$g = g_1^{m_1} g_2^{m_2} \cdots g_k^{m_k}$$

where $g_i \in D = \{b^{\pm 1}, f_0^{\pm 1}, f_1^{\pm 1}\}$, $m_i > 0$ $(i = 1, \ldots, k)$, and $g_i \neq g_{i+1}^{\pm 1}$ $(i = 1, \ldots, k-1)$. Then g is (λ, ε)-quasigeodesic in G.

Corollary 2. *Let G be a non-elementary hyperbolic group. Let words b, f_0, f_1 be such that elements $\overline{b}, \overline{f}_0, \overline{f}_1 \in G$ are pairwise non-commensurable. Suppose*

$$\overline{g}_1^{m_1} \overline{g}_2^{m_2} \cdots \overline{g}_k^{m_k} = \overline{g'_1}^{m'_1} \overline{g'_2}^{m'_2} \cdots \overline{g'_l}^{m'_l},$$

where $g_i, g'_i \in D = \{b^{\pm 1}, f_0^{\pm 1}, f_1^{\pm 1}\}$, $m_i, m'_i > 0$, and $g_i \neq g_{i+1}^{\pm 1}$, $g'_i \neq (g'_{i+1})^{\pm 1}$. Then $k = l$ and $g_i = g'_i$.

Proof. Apply Corollary 1 to the product

$$(g_k^{-1})^{m_k} \cdots (g_2^{-1})^{m_2} (g_1^{-1})^{m_1} {g'_1}^{m'_1} {g'_2}^{m'_2} \cdots {g'_l}^{m'_l}.$$

$\square$

Proposition 2. *Let G be a non-elementary hyperbolic group. Let words b, f_0, f_1 be such that elements $\overline{b}, \overline{f}_0, \overline{f}_1 \in G$ are pairwise non-commensurable and let constants λ, ε be as provided in Corollary 1. Let $K \in \mathbb{Z}$, $K > 0$ be given. Then there exists $M \in \mathbb{Z}$ such that the following condition holds. Let $g \in G$ be such that*

$$g = \overline{b}_1 \overline{g}_1^{m_1} \overline{g}_2^{m_2} \cdots \overline{g}_k^{m_k} \overline{b}_2 = \overline{b}_3 \overline{g'}_1^{m'_1} \overline{g'}_2^{m'_2} \cdots \overline{g'}_l^{m'_l} \overline{b}_4$$

where $|b_i| \leq K$, $g_i, g'_i \in D = \{b^{\pm 1}, f_0^{\pm 1}, f_1^{\pm 1}\}$, $m_i \geq M$ for $i = 2, \dots, k-1$, $m'_i \geq M$ for $i = 2, \dots, l-1$, and $g_{i-1} \neq g_i^{\pm 1}$, $g'_{i-1} \neq {g'_i}^{\pm 1}$. Then one of the following options occurs:

(a) $(g_2, g_3, \dots, g_{k-1}) = (g'_2, g'_3, \dots, g'_{l-1})$, *or*
(b) $(g_2, g_3, \dots, g_{k-1}) = (g'_2, g'_3, \dots, g'_l)$, *or*
(c) $(g_2, g_3, \dots, g_{k-1}) = (g'_1, g'_3, \dots, g'_{l-1})$, *or*
(d) $(g_2, g_3, \dots, g_{k-1}) = (g'_1, g'_3, \dots, g'_l)$.

Proof. By Lemma 4, $g_1^{m_1} \cdots g_k^{m_k}$ and ${g'_1}^{m'_1} \cdots {g'_l}^{m'_l}$ are (λ, ε)-quasigeodesics. Then $b_1 g_1^{m_1} \cdots g_k^{m_k} b_2$ and $b_3 {g'_1}^{m'_1} \cdots {g'_l}^{m'_l} b_4$ are (λ, ε_0)-quasigeodesics, where ε_0 depends of λ, ε, K. By Lemma 3, paths corresponding to these two words asynchronously L-fellow travel for some $L = L(\delta, \lambda, \varepsilon, K)$.

If M is larger than the number of words of length $(L + \max\{|b|, |f_0|, |f_1|\})$, asynchronous fellow travel property guarantees that g_2 and g'_1, or g_1 and g'_2, or g_2 and g'_2 are commensurable, thus equal. The statement then follows by an inductive argument. $\square$

In a non-elementary hyperbolic group G it is possible to find elements $\overline{b}, \overline{f}_0, \overline{f}_1 \in G$ that are pairwise non-commensurable. For the rest of the paper, fix such elements and the set $D = \{b^{\pm 1}, f_0^{\pm 1}, f_1^{\pm 1}\}$. Let $M \in \mathbb{Z}$ be as described in Proposition 2. Then we say that (b, f_0, f_1, M) is a *big powers tuple*. Further, an expression of the form

$$g_1^{m_1} g_2^{m_2} \cdots g_k^{m_k}, \tag{1}$$

where $g_i \in D$, $m_i \geq M$ is termed a (b, f_0, f_1, M) *big powers product*.

Define a set of elements $R = R(b, f_0, f_1, M) \subseteq G$ to be

$$\begin{aligned} R = \{g \in G \mid \exists k \in \mathbb{N}, g_i \in D, m_i \geq M,\ g_{i-1} \neq g_i^{\pm 1}, \\ g = \overline{g}_1^{m_1} \overline{g}_2^{m_2} \cdots \overline{g}_k^{m_k}\}, \end{aligned}$$

that is the set of elements representable in the form (1).

4. Counting gaps

For a big powers product g, its factor of the form

$$b^{m_\mu} g_{\mu+1}^{m_{\mu+1}} \cdots g_{\mu+\nu}^{m_{\mu+\nu}} b^{m_{\mu+\nu+1}},$$

where $g_i \neq b, b^{-1}$, is called a b-*syllable*. We define b^{-1}-syllables similarly. With each b-syllable s we associate an integer $\omega_s \in \mathbb{Z}$ so that

$$\omega_s = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_\nu,$$

where $\varepsilon_i = 0$ if $g_{\mu+i} = f_1^{\pm 1}$, and ε_i is such that $g_{\mu+i} = f_0^{\varepsilon_i}$, otherwise.

ω_s is called the b-*gap* associated with s. We define b^{-1}-gaps similarly. Following Rhemtulla [38], given $\omega \in \mathbb{Z}$ and $\overline{x} \in R$ we define $\delta_\omega(x)$ to be the number of b-gaps ω in x. By Proposition 1, $\delta_\omega(x)$ is well-defined. By $\delta^*_\omega(x)$ we denote the number of b^{-1}-gaps ω. Observe that

$$\delta_\omega(x) = \delta^*_{-\omega}(x^{-1}).$$

Further, by $\gamma(x)$ we denote the number of integers $\omega \in \mathbb{Z}$ such that

$$\delta_\omega(x) - \delta^*_{-\omega}(x) \neq 0 \mod d$$

(in particular, if $d = 0$, by this inequality transforms into $\delta_\omega(x) - \delta^*_{-\omega}(x) \neq 0$ as integers). Note that

$$\gamma(x) = \gamma(x^{-1}).$$

Denote $\delta_\omega(x) - \delta^*_{-\omega}(x)$ by $\Delta_\omega(x)$. By Corollary 2, $\delta_\omega, \delta^*_\omega, \Delta_\omega, \gamma$ are also well defined as functions of $g \in R$.

We will show that for each $w \in F$, $e(w) = d$ and for each l there is an appropriate choice of b, f_0, f_1, M such that γ is bounded on $w[G]^l \cap R$ and unbounded on $w(G) \cap R$.

Remark. Note that in general, it is not true that if $xy \in R$, then $x \in R, y \in R$.

Lemma 5. *Let $p_1 p_2 \dots p_m$ be a (λ, ε)-quasigeodesic m-gon in a δ-hyperbolic space. Then there is a constant $H = H(\delta, \lambda, \varepsilon, m)$ such that each side p_i, $1 \leq i \leq m$, belongs to a closed H-neighborhood of union of other sides.*

Proof. It suffices to prove the case $i = 1$. In fact, we will prove the following statement (see Fig. 1): there exist points $Q_1, \dots, Q_k$ on p_1 and points P_{i1}, P_{i2} on $p_2 \dots p_m$, $1 \leq i \leq k$, such that each $|Q_i P_{ij}| \leq H$, $|P_{i1} P_{i2}| \leq H$, $P_{11} \in p_2$, $P_{k2} \in p_n$, and $P_{i2}, P_{i+1,1}$ belong to the same p_{k_i}. The required statement then follows by Lemma 3.

We use induction in m. Case $m = 3$ is evident. For $m > 3$, we assume that H is such that both original and the restated version of the lemma hold for $\leq (m-1)$-gons.

Let P_1 denote the common vertex of p_1, p_2. Draw a geodesic diagonal q_{23} joining P_1 with P_3, the terminus of p_3. Using the induction hypothesis for the $(m-1)$-gon $p_1 q_{23} p_4 \dots p_m$, obtain points $Q_1, \dots, Q_k \in p_1$. Using the induction hypothesis for the triangle $q_{23} p_2 p_3$, obtain points $Q_0 \in q_{23}$,

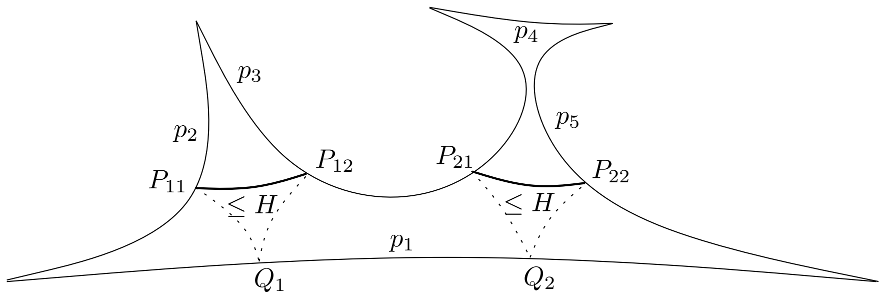

FIGURE 1. Restatement of Lemma 5.

$P_{01} \in p_2$, $P_{02} \in p_3$. There are two cases, whether Q_0 lies on the arc P_1P_{11} of q_{23}, or on the arc $P_{11}P_3$ of q_{23}.

Case 1. $Q_0 \in P_1P_{11}$. (This case is illustrated by Fig. 2.) Then by the induction hypothesis, there is a point $Q'_0 \in p_1$ such that $|Q_0Q'_0| \leq H$, and a point $P'_{11} \in p_3$ such that $|P_{11}P'_{11}| \leq H$. The statement is then delivered by points $Q'_0, Q_1, \ldots, Q_k$ and $P_{01}, P_{02}, P'_{11}, P_{12}, P_{21}, \ldots, P_{k2}$.

Case 2. $Q_0 \in P_{11}P_3$. In this case, by the induction hypothesis there is a point $P'_{11} \in p_2$ such that $|P_{11}P'_{11}| \leq H$. The statement is then delivered by points $Q_1, \ldots, Q_k$ and $P'_{11}, P_{12}, P_{21}, \ldots, P_{k2}$. □

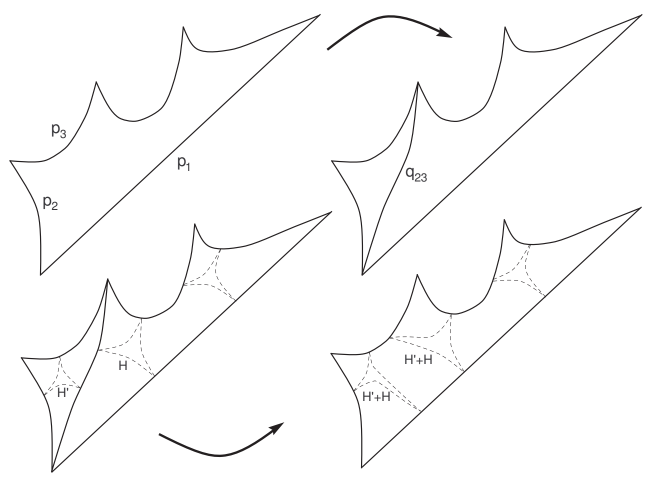

FIGURE 2. Case 1 in the proof of Lemma 5.

Lemma 6. *Suppose $w = w(x_1, \ldots, x_n) \in F$ and $d = e(w)$. Suppose $h_1, h_2, \ldots, h_n \in F$ and $w(h_1, \ldots, h_n) = g \in F$. Then there exist $y_1, y_2, \ldots, y_s \in F$ such that $g = y_{i_1}^{\delta_1} y_{i_2}^{\delta_1} \cdots y_{i_N}^{\delta_N}$, where N is bounded by a function of w, there*

is no cancelation between any $y_{i_j}^{\delta_j}$ *and* $y_{i_{j+1}}^{\delta_{j+1}}$, *and each* y_ν *occurs a number of times divisible by* d *(counting every occurrence of* y_ν^{-1} *as* -1*), i.e.* $\sum\limits_{y_{i_j}=y_\nu} \delta_j = dp$ *for some* $p \in \mathbb{Z}$.

Proof. (The proof is illustrated by Figure 3.) Let $w = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_r}^{\epsilon_r}$, $\epsilon_i = \pm 1$. Then in the word $h_{i_1}^{\epsilon_1} h_{i_2}^{\epsilon_2} \cdots h_{i_r}^{\epsilon_r} g^{-1}$, each letter of each $h_{i_j}^{\epsilon_j}$ cancels with a letter of g or of some $h_{i_{j'}}^{\epsilon_{j'}}$. According to this cancelation arrangement, split each h_{i_j} as a product

$$h_{i_j} = h_{i_j}^{(1)} \cdot h_{i_j}^{(2)} \cdots h_{i_j}^{(I_{i_j})}, \tag{2}$$

where $I_{i_j} \le r+1$.

Further, split every h_ν as a product $h_\nu = h_\nu^{[1]} \cdots h_\nu^{[J_\nu]}$ that is obeyed by each splitting (2) with $h_{i_j} = h_\nu$, that is, for every j and ν such that $h_{i_j} = h_\nu$, and every $1 \le \mu \le I_{i_j}$ we have

$$h_{i_j}^{(\mu)} = h_\nu^{[K_{i_j}]} \dots h_\nu^{[L_{i_j}]}.$$

Note that we may assume $J_\nu \le (r+1)^2$.

Denote

$$\{y_1, \dots, y_s\} = \{h_\nu^{[\eta]} \mid 1 \le \nu \le k, 1 \le \eta \le J_\nu\}.$$

Then each y_i occurs in $g = w(h_1, \dots, h_k) = y_{\xi_1}^{\delta_1} y_{\xi_2}^{\delta_2} \cdots y_{\xi_L}^{\delta_L}$, $\delta_i = \pm 1$, (counting opposite oriented occurrences) dp times. Performing free cancelations in $y_{\xi_1}^{\delta_1} y_{\xi_2}^{\delta_2} \cdots y_{\xi_L}^{\delta_L}$ (as a word in a free group generated by y_i), we obtain the required representation $g = y_{i_1}^{\delta'_1} y_{i_2}^{\delta'_2} \cdots y_{i_N}^{\delta'_N}$, $\delta'_i = \pm 1$, $N \le L \le (r+1)^3$. Note that canceled out pieces y_i come in opposite oriented pairs, so after the cancelation, each y_i still occurs dp' times (counting opposite oriented occurrences). $\square$

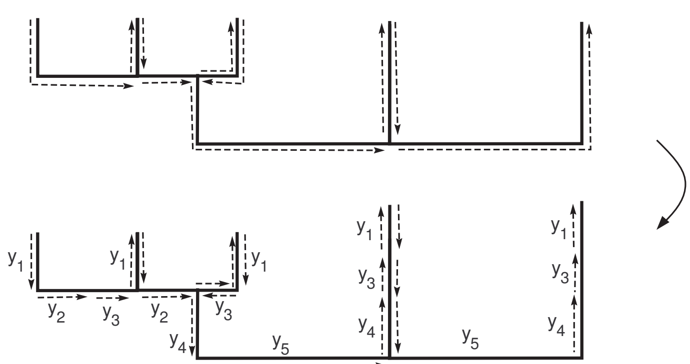

FIGURE 3. $h_1^2 h_2^2 = y_1 \cdot y_2 \cdot y_3 \cdot y_2 \cdot y_4 \cdot y_5 \cdot y_5 \cdot y_4^{-1} \cdot y_3^{-1} \cdot y_1^{-1}$.

Proposition 3. *Suppose $w \in F$ with $d = e(w)$. Then $\exists M \in \mathbb{Z}$, $\exists N \in \mathbb{Z}$ s.t. if $g \in w[G] \cap R(b, f_0, f_1, M)$, then $\gamma(g) \leq N$.*

Proof. Suppose $g \in G$ is such that $g = w(\overline{h}_1, \overline{h}_2, \dots, \overline{h}_n)$, $\overline{h}_i \in G$. Apply Lemma 5 to the polygon $p_g^{-1} w(h_1, \dots, h_n)$, where h_i are geodesic words and p_g is a (λ, ε)-quasigeodesic big powers product corresponding to $g \in R$. Then p_g is split into segments so that each segment asynchronously fellow travels with a segment of one of $h_1, h_2, \dots, h_n$. Specifically, we obtain a constant H (by Lemma 3, H depends on $\delta, \lambda, \varepsilon$ and the word $w \in F$) such that geodesic words $h_1, \dots, h_n$ split into subwords $z_1, \dots, z_r$ (see Fig. 4) that provide the following $(\lambda'', \varepsilon'')$-quasigeodesic representation for g:

$$g = \overline{z_{i_1} d_1 z_{i_2} d_2 \dots d_{N_0 - 1} z_{i_{N_0}}},$$

where $|d_i| \leq H$, $(\lambda'', \varepsilon'')$ depend on $\delta, \lambda, \varepsilon$ and the word w, and paths that correspond to words z_{i_j} asynchronously H-fellow travel with segments of p_g.

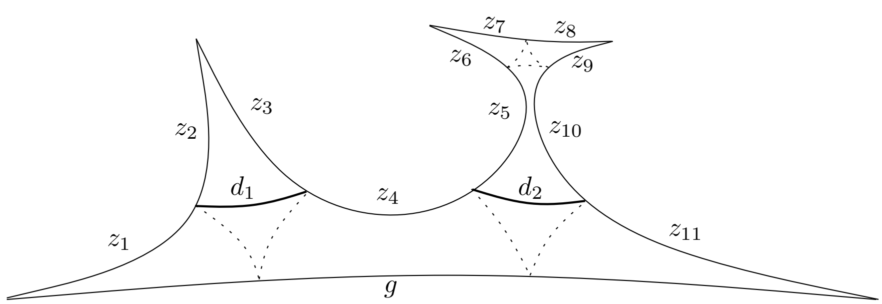

FIGURE 4. $g = \overline{z_1 d_1 z_4 d_2 z_{11}}$.

Identifying fellow traveling segments of $h_1, \dots, h_n, p_g$ (see Fig. 5) and applying Lemma 6 to the obtained diagram, we further subdivide $z_1, \dots, z_r$ into subwords $y_1, \dots, y_s$ so that $z_j = y_{j_1}^{\delta_1} \cdot \dots \cdot y_{j_{K_j}}^{\delta_{K_j}}$, $\delta_i = \pm 1$, and words y_i split into classes $Y_1, Y_2, \dots, Y_t$ such that in each class $Y_i = \{y_{\nu_1^i}, \dots, y_{\nu_{N_i}^i}\}$, each pair $y_{\nu_j^i}, y_{\nu_{j+1}^i}$ is either a pair of equal words, or a pair of H-fellow traveling words; so arbitrary words $y_{\nu_j^i}, y_{\nu_{j'}^i}$ in the same class Y_i asynchronously $(N_i H)$-fellow travel, $N_i H \leq sH$. Therefore, picking one representative for each Y_i, we obtain

$$g = \overline{y_{j_1} c_1 y_{j_2} c_2 \dots c_{N-1} y_{j_N}},$$

where $|c_i| \leq |d_i| + 2sH \leq (1 + 2s)H = H'$, the above representation is (λ', ε')-quasigeodesic with λ', ε' depending on $\delta, \lambda, \varepsilon$ and the word w, each path corresponding to y_{j_i} asynchronously H'-fellow travels with a piece of p_g. Finally, by Lemma 6, each y_i occurs dp times (counting opposite oriented occurrences), and N and s are bounded by a function of w.

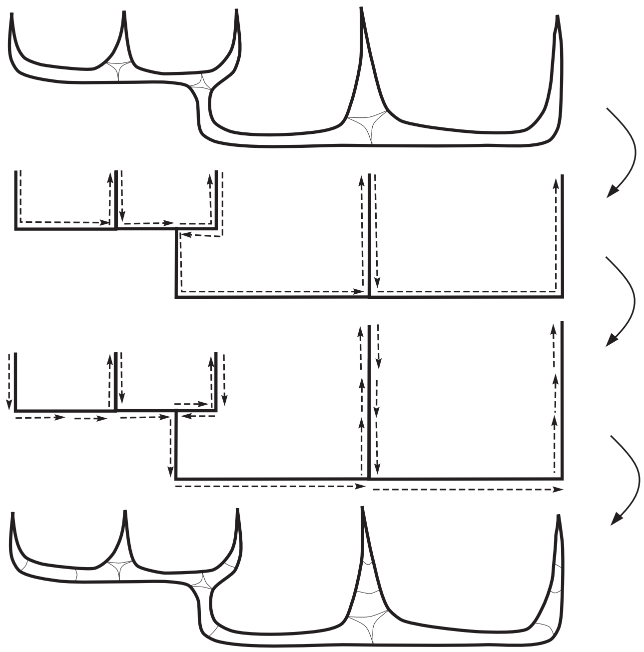

FIGURE 5. Splitting $g = w(\overline{h}_1, \ldots, \overline{h}_k)$ into fellow traveling pieces.

Recall that $g \in R$, so g is represented by two quasigeodesics

$$g_1^{m_1} g_2^{m_2} \cdots g_q^{m_q},$$
$$y_{i_1} c_1 y_{i_2} c_2 \ldots c_{N-1} y_{i_N}$$

with parameters (λ, ε) and (λ', ε'), respectively. By Lemma 3, they asynchronously K-fellow travel for some $K = K(\delta, \lambda, \lambda', \varepsilon, \varepsilon')$.

If $y_{i_\mu} = y_{i_\nu}$ (the case $y_{i_\mu} = y_{i_\nu}^{-1}$ is treated similarly), then the corresponding (under the asynchronous fellow travel obtained above) segments of $g_1^{m_1} g_2^{m_2} \cdots g_q^{m_q}$ are also equal (see Fig. 6):

$$\overline{y} = \overline{y_{i_\mu}} = \overline{b_1 g_1' (g_{j_\mu - 1}^{m'_{j_\mu - 1}} g_{j_\mu}^{m_{j_\mu}} \cdots g_{k_\mu}^{m'_{k_\mu}}) g_2' b_2} =$$
$$\overline{y_{i_\nu}} = \overline{b_3 g_3' (g_{j_\nu - 1}^{m'_{j_\nu - 1}} g_{j_\nu}^{m_{j_\nu}} \cdots g_{k_\nu}^{m'_{k_\nu}}) g_4' b_4},$$

where $|b_i|$ are bounded by some $K(\delta, \lambda, \varepsilon)$ (by Lemma 3), so $|b_i g_i'|$, $i = 1, 3$, and $|g_i' b_i|$, $i = 2, 4$, are bounded by $K + \max\{|b|, |f_0|, |f_1|\}$.

By Proposition 2, an appropriate choice of M guarantees that sequences $(g_{j_\mu - 1}, g_{j_\mu}, \ldots, g_{k_\mu})$ and $(g_{j_\nu - 1}, g_{j_\nu}, \ldots, g_{k_\nu})$ are equal with possible exception to the initial and final terms.

Therefore, for any b-gap ω whose associated b-syllables are contained in segments $g_{j_\mu}^{m_{j_\mu}} \cdots g_{k_\mu - 1}^{m_{k_\mu - 1}}$, we have $\Delta_\omega(g) = 0 \mod d$. The number of "boundary" syllables is bounded by $3N$, where N is described in Lemma 6.

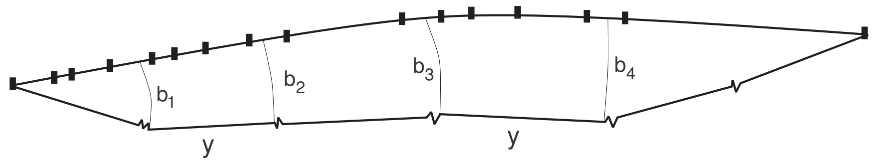

FIGURE 6. $\overline{g_1^{m_1} g_2^{m_2} \cdots g_q^{m_q}} = \overline{y_{i_1} c_1 y_{i_2} c_2 \ldots c_{N-1} y_{i_N}}$.

Since c_i are bounded in length by H', an appropriate choice of M provides that the number of syllables that correspond (under fellow travel) to c_i, is bounded by $3(N-1)$.

This shows $\gamma(g) \leq 6N$.

$\square$

5. Proof of Theorem 1

For a word $w = w(x_1, \ldots, x_n)$, by $W^{(l)} \subset F$ we denote the following finite set of words in variables $x_1, \ldots, x_{nl}$:

$$W^{(l)} = \{w^{\pm 1}(x_1, \ldots, x_n) w^{\pm 1}(x_{n+1}, \ldots, x_{2n}) \cdots \cdot w^{\pm 1}(x_{(l-1)n+1}, \ldots, x_{(l-1)n+n})\}.$$

Note that $w[G]^l = \cup_{u \in W^{(l)}} u[G]$, and $e(u) = e(w)$ for every $u \in W^{(l)}$. For a fixed l, applying Proposition 3 to all words in $W^{(l)}$, we conclude that for $g \in R \cap w[G]^l$, $\gamma(g)$ is bounded by a number N that depends on w and l. It is only left to note that one can easily point out a sequence of elements h_k in $R \cap w(G)$ with $\gamma(h_k) \to \infty$. (From now on, with an obvious abuse of notation, we don't keep track of difference between words b, f_0, f_1 in $\mathcal{A}$ and respective elements $\overline{b}, \overline{f}_0, \overline{f}_1$ of G.)

Case 1. Indeed, let $d = e(w) > 0$. Denote

$$X_j = f_1^M f_0^M f_1^M \cdots f_0^M f_1^M \cdot b^M \cdot f_1^M f_0^M \cdots f_1^M f_0^M,$$

where f_0^M is repeated j times on the left of b^M and j times on the right of it. Then put

$$h_k = X_1^d X_2^d \cdots X_k^d.$$

$h_k \in R \cap w(G)$ since it is a product of d-th powers. Note that $\Delta_\omega(h_k) = 1$ for every odd number between 3 and $2k-1$ (and $\Delta_\omega(h_k) = -1 \mod d$ for every even number between 2 and $2k$). Therefore, $\gamma(h_k) \geq k - 2 \to \infty$.

Case 2. Now, suppose $d = e(w) = 0$. Renaming and/or inverting variables and passing to a conjugate, we may assume $w(x_1, x_2, \ldots, x_n) = x_1 \cdots x_2$. Consider the following elements of the group G:

$$X_{ij} = f_0^{inM+(j-1)M} f_1^M f_0^{inM+(j-1)M}, \quad 1 \leq i < \infty,\ 1 \leq j \leq n,$$

and

$$Y_j = b^{jM} f_1^M b^{jM}, \quad 1 \leq j \leq n.$$

Note that by Corollary 2, $\{X_{ij}, Y_j \mid 1 \leq i < \infty,\ 1 \leq j \leq n\}$ is an infinite free basis for a free subgroup of G. Consider element $g_1 \in G$ defined by

$$g_1 = w(X_{11}^{-1}, X_{12}, \ldots, X_{1n}) = f_0^{\alpha_1} f_1^{\beta_1} f_0^{\alpha_2} \cdots f_1^{\beta_s} f_0^{\alpha_{s+1}}.$$

Note that, since w is a commutator word, s is even, and therefore $s+1$ is odd, so the value

$$\varepsilon(g_1) = \varepsilon_1 + \cdots + \varepsilon_{s+1}$$

is nonzero, where $\varepsilon_\mu = 1$ if $\alpha_\mu > 0$ and $\varepsilon_\mu = -1$ otherwise. Note that in any case $\alpha_1 < 0$ and $\alpha_{s+1} > 0$. Put

$$g_2 = \begin{cases} w(X_{21}^{-1}, X_{22}, \ldots, X_{2n}) g_1, & \text{if } \varepsilon(g_1) > 0; \\ g_1 w(X_{21}^{-1}, X_{22}, \ldots, X_{2n}), & \text{if } \varepsilon(g_1) < 0. \end{cases}$$

Note that $\varepsilon(w(X_{21}^{-1}, X_{22}, \ldots, X_{2n})) = \varepsilon(g_1)$, so $|\varepsilon(g_2)| \geq 2|\varepsilon(g_1)| > |\varepsilon(g_1)|$. Proceed in the same fashion: if $i \geq 3$ and

$$g_{i-1} = f_0^{\alpha'_1} f_1^{\beta'_1} f_0^{\alpha'_2} \cdots f_1^{\beta'_{s'}} f_0^{\alpha'_{s'+1}}$$

with $\alpha'_1 < 0$, $\alpha'_{s'+1} > 0$, we put

$$g_i = \begin{cases} w(X_{i1}^{-1}, X_{i2}, \ldots, X_{in}) g_{i-1}, & \text{if } \varepsilon(g_{i-1}) > 0; \\ g_{i-1} w(X_{i1}^{-1}, X_{i2}, \ldots, X_{in}), & \text{if } \varepsilon(g_{i-1}) < 0. \end{cases}$$

Then the initial power of f_0 in g_i is negative, the terminal one is positive, and $\varepsilon(w(X_{i1}^{-1}, X_{i2}, \ldots, X_{in})) = \varepsilon(g_1)$, so $|\varepsilon(g_i)| > |\varepsilon(g_{i-1})|$.

Finally, denote $w(Y_1, \ldots, Y_n) = B$ and consider elements of G defined by

$$h_k = B g_1 B g_2 B \ldots B g_k B \in w(G) \cap R.$$

Note that $\Delta_{\varepsilon(g_i)}(h_k) = 1$ for any $1 \leq i \leq k$. Therefore, $\gamma(h_k) \geq k \to \infty$.

This finishes the proof of Theorem 1.

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